Efficiency with natural resources

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Abstract

This paper is to study the Pareto e ciency of the equilibrium allocation in OLG models with three production factors: physical capital, labor and natural resource. We present general su cient conditions and general necessary conditions for the Pareto e ciency of the equilibrium allocation. Then, the general results are applied to two cases: the natural resource regeneration function is linear or quadratic, respectively. In the linear case (with additive log utility function and CES production function), we prove that there are a continuum of steady state equilibria, among which, some are Pareto e cient, some are not, depending on the speed of the resource harvesting, the more slowly, the more prone to be Pareto e cient. In the quadratic case (with additive log utility function and Cobb-Douglas production function), we prove that there is a unique steady state equilibrium, and there is an aggregate capital index (combining the physical capital and the natural capital), if the labor share is smaller than this index, then, the equilibrium allocation is Pareto e cient; if the labor share is bigger than this index, then, it's Pareto ine cient.

JEL classification: O13; O41; Q20; Q32

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1 Introduction

In resource economics, OLG model is one of the typical workhorses in analyzing the natural resource usage. It's well known that in the OLG model, even without public goods, externality, trade friction, market power, the issue of incomplete information, and the issue of non-convexity, etc, the equilibrium allocation may or may not be Pareto e cient, so that the rst welfare theorem may fail, in other words, Adam Smith's invisible hand (market mechanism)

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may not work. The resources of inefficiency are thought to include, say, double infinity, intertemporal trade barrier, etc.

Finding a general powerful criterion to judge the Pareto efficiency of the equilibrium allocation is meaningful. Through such a criterion, we can reveal the concrete key elements in the efficiency issue in various situations. Its study dates back to the fifties of the last century. There is a popular view: the growth rate of investment vs the rate of return of investment, if the former is higher, then, inefficient, capital is over-accumulated; if the later is higher, then, efficient. But, what does the word "investment" mean exactly? If treated rigorously, especially in the environment with natural resources, it becomes a very tough problem. Many papers are devoted to it. We first give a quick review of the literature without natural resources.

Balasko and Shell (1980, 1981a, 1981b) study an OLG pure exchange economy, under some setup, present a Cass type sufficiency condition (see below) for a feasible allocation to be Pareto efficient. Wilson (1981), in general OLG economies, presents a sufficiency condition, something like the sum of all discounted outputs is convergent. Abel, Mankiw, Summers, and Zeckhauser(1989)(AMSZ(1989), for short), a seminal paper, in a general random environment setting, "proves" the popular view described above. But the proof of the sufficiency part in AMSZ(1989) is not rigorous. Chattopadhyay (2008) constructs a counterexample to show that the sufficiency part of AMSZ(1989) is not correct[1]. Geanakoplos et al (1991)[2], in an OLG pure exchange economy, presents a sufficient condition that the first generation owns a resource contributing to income with a strictly positive share bounded away from zero in every period. Homburg (1992) (Theorem 1) presents a sufficiency condition, meaning that if there remains nothing valuable at the end of time, then, the equilibrium allocation is Pareto efficient. Croix et al (2004) (Lemma 2.1), in the standard Diamond OLG model (Diamond (1965)), presents a sufficiency condition that the discounted wages converge to 0. Tirole (1985), in an OLG model with productive and nonproductive assets, in part, demonstrates various equilibria, some are Pareto efficiency, some are not, it relates to the limit interest rate and the bubbles. Drugeon and Venditti (2010) considers the efficiency problem for a two-sector OLG model, but this efficiency is the so-called dynamic efficiency (see below) but not Pareto efficiency, these two concepts are different, and in general, the issue of Pareto efficiency is quite more difficult than the issue of dynamic efficiency.

A related problem exists in the so-called Ramsey economy, the simplest form of which is that there is one-sector, and there is a representative agent with infinite life[3]. For such a Ramsey economy, the issue of dynamic efficiency of any feasible allocation is studied intensively in the literature.

The most famous works are Malinvaud sufficiency condition (the present

[1] See also Miao(2020).

[2] see also Geanakoplos(2008), which gives an excellent survey of literature on OLG.

[3] The more general form is that there are heterogeneous agents with infinite life. See Becker R.A., Mitra T. (2012). In such a setting, the competitive equilibrium may also be Pareto inefficient.

value of capital converges to zero) (Malinvaud (1953)), Cass criterion (the sum of the reciprocals of the norms of prices is divergent) (Cass(1972)). Benveniste and Gale D.(1975) extends Cass criterion to a more general case. Mitra (1979) gives a necessary and su cient condition for a feasible allocation to be dynamically ine cient in a quite general setting, including the works of Malinvaud and Benveniste and Gale as special cases.

An OLG economy can be \contracted" to a Ramsey economy by aggregating all the people to one representative agent. Then, any feasible allocation in the OLG economy corresponds to a feasible program in the corresponding Ramsey economy. The two concepts of e ciency: Pareto e ciency in OLG economy and the dynamic e ciency in Ramsey economy are related but di erent. For more detail, see section 2.

In the Ramsey economy framework, Mitra(1978) presents a Malinvaud type su ciency condition for a feasible program in a Ramsey economy with exhaustible resources to be dynamically e cient: the total present value of all assets goes to zero.

Concerning the OLG economy with natural resources, the relevant literature is as follows.

Rhee (1991) (Proposition 1) shows that an OLG economy with land will be Pareto e cient if the income share of land does not vanish in the long run. Mourmouras (1991) considers such an OLG model with a natural resource owning a linear regeneration function and physical capital, but, he only investigates the problem that how high the regeneration capability of this natural resource is needed in order to get sustainability, he does not discuss the Pareto e ciency problem. Olson et al. (1997) proves the Pareto e ciency property of the competitive equilibrium in an OLG model with exhaustible resources but without physical capital, the factors of production are only natural resource and labor. Krautkraemer (1999) considers an OLG model with a natural resource owning logistic regeneration function but without physical capital, he focuses on the problem of the existence of steady-state equilibrium and its properties, including the Pareto e ciency problem of the equilibrium. Farmer(2000) considers an OLG model with a natural resource owning logistic regeneration function and physical capital, he also focuses on the problems of the existence of steady-state equilibria and sustainability, he does not discuss the e ciency problem of the equilibrium. In addition, he does not consider the technical change and population growth. Koskela, et al (2002) extends Olson et al. (1997) and considers a two-period OLG model with a natural resource but without physical capital, the utility function is quasi-linear, the regeneration function of the natural resource is logistic, and discusses the problems of the existence and the e ciency of steady-state equilibria. Betty, et al. (2005) discuss sustainability issue and the Pareto e ciency problem in an OLG model with exhaustible resource and physical capital, but, they do not prove the existence (and uniqueness) of the equilibrium, and assume the existence of the equilibrium and assume furthermore that the equilibrium path is a balanced growth path, under all of those assumptions, they prove that the equilibrium is socially optimal. Farmer et al. (2017) extend Koskela, et al (2002) in the sense that they consider various harvest costs, but the individual utility function is additive log function, the production function is of Cobb-Douglas without physical capital, the natural resource regeneration function is logistic function, they discuss the existence and the efficiency (or inefficiency) of stationary equilibrium.

In this paper, we consider an OLG economy with three factors of production: physical capital, labor and a natural resource, and present general sufficiency conditions and general necessary conditions for the equilibrium allocation to be Pareto efficient, either in terms of wage, or in terms of capital, or total value of assets. In particular, our necessary condition in terms of wage is new in the literature, to our knowledge: the present values of wages can not be increasing. The basic approach of our proof is to construct a Pareto improvement by transferring consumption from young to old. It's analogous to the issue of Hilbert's Hotel, but is more complicated.

Then, the main results will be applied to two special cases: the natural resource regeneration function is linear or quadratic. For the linear case with additive log utility function and Cobb-Douglas production function, we give a rigorous proof of the existence and uniqueness of the equilibrium, and prove the Pareto efficiency of the equilibrium allocation without any additional condition. For the linear case with additive log utility function and general CES production function, we demonstrate that there are a continuum of steady state equilibria, some are Pareto efficient, some are not, roughly, the more slowly the resource is harvested, the better in the sense that it's more prone to be Pareto efficient. For the quadratic case (with additive log utility function), we give a proof of the existence and uniqueness of the equilibrium, and find that the labor share plays a crucial role, and there is an aggregate capital index combining the physical capital share and the the natural capital share such that if the labor share is smaller than this index, then, the equilibrium allocation is Pareto efficient, if the labor share is bigger than this index, than, the equilibrium allocation is inefficient.

All the results in these examples coincide with the popular view, which is "proved" in AMSZ(1989). But, we need to point out some features of AMSZ(1989). Although they treat the efficiency issue in a random environment, but, randomness does not induce any difficulties. In addition, in their main results, they need the rate of return of investment is either always bigger than or always smaller than the growth rate of investment. Just as they mentioned in their paper (AMSZ(1989) pp.12): *Neither implication is very helpful in judging the dynamic efficiency of actual economies, where capital gains and losses cause the growth rate of the market value of the capital stock sometimes to exceed and other times to fall short of the safe interest rate. The result here is illuminating primarily in suggesting that comparisons of the safe interest rate with the average growth rate generally are not sufficient to resolve the issue of dynamic efficiency.* In this paper, for the necessity part, we only need to compare them in the limit but not the whole process of the economic development. As to the sufficiency part, due to the counterexample in Chattopadhyay (2008), we take a different approach. In addition, we notice that in AMSZ(1989), the production is based only on factors such as the savings from human production (rest of outputs, after subtracting consumptions) in some previous periods, besides labor and some random perturbation, which do not include natural resources, which are "saved" by natural and people jointly from the pervious periods (For more detail, see section 3). Therefore, AMSZ(1989) can not cover the case with natural resources.

In general, the resources of inefficiency mentioned above, say, double infinity, intertemporal trade barrier, are not cleaned up by adding natural resources to the OLG economy. It needs some other interventions, e.g. from government. This will be another problem. We do not discuss it in this paper.

The structure od this paper is as follows. In section 2, we set up the model framework, and give definitions of some fundamental concepts. In section 3, we present our main results. The following two sections are devoted to linear and quadratic cases. In section 6, we make the conclusion.

For simplicity, throughout this paper, for any sequence of positive numbers $(x_t)_{t=0,1,2,\ldots}$, we use the notation $x_t =: x_{t+1}/x_t$. Then, $x_t - 1$ is its growth rate.

2 Model setup

Consider a closed two-period competitive OLG economy, which exists in the time points $t \in \mathbb{N} = \{0, 1, 2, \ldots\}$.

Population.

At any time $t \in \mathbb{N}$, there are $N_t > 0$ homogeneous young men, who become old at $t + 1$, and dies after then. At time $t = 0$, there is the original generation with population $N_{-1} > 0$, each one of which is as old at $t = 0$ and dies after then.

Endowments.

At any time $t \in \mathbb{N}$, each one of the young men is only endowed with one unit of labor, and all the old men of the original generation share evenly the physical capital $K_0 > 0$ and natural resource $S_0 > 0$. (From now on, K_0 and S_0 are fixed constants, and are reserved for these meanings throughout this paper)

Firms.

At any time $t \in \mathbb{N}$, there is only one sector, there are many but finite homogeneous firms with the same technology, the production function of which is

$$Y = F^t(K, L, R),$$

where Y is the output of the final good, and K, L, R are the inputs of factors: the physical capital, labor and natural resource, respectively, F^t is a first order homogeneous function, smooth and concave. The final good can be used for consumption as well as investment in physical capital. Each firm exists only one period.

Utility function.

For any time $t \in \mathbb{N}$, anyone of t-generation has a utility function $U(a_t, b_{t+1})$, where a_t and b_{t+1} are his consumptions at t and $t + 1$, respectively, and U is strictly increasing wrt to each element, smooth and concave. And the utility function for anyone of the original generation (ancestor) is $u(b_0)$, where u is strictly increasing, smooth and concave, and b_0 is her consumption at $t = 0$.

Natural resource.

The natural resource, as a single entity (although it may be changed by nature), is harvested and sold, it itself is not divided physically among the owners. The owners share evenly the property rights on this natural resource, and hence share evenly the revenue from it[1]. The natural resource harvesting is costless.

Concerning the transaction and the motion law of the natural resource, there are two typical approaches.

Approach one. For any $t \in \mathbb{N}$, at the beginning of period t, the natural resource, the stock of which is S_t, is owned by all the old men, it is sold to the young men, the property rights of the natural resource is transfered from the old to the young. And then, a part of the natural resource, R_t, is harvested and sold to the firms by the young men, we suppose that in the process of the production of the final good, the natural resource is depreciated by the depreciation rate $d \in [0; 1]$, therefore, after use for production, the firms return to the young men the quantity of natural resource $(1 - d)R_t$, and hence, the rest stock of the resource is $S_t - dR_t$, which is held by the young men and grows to $G(S_t - dR_t)$ at the beginning of period $t+1$. Concerning the depreciation rate of the natural resource, two extreme cases are: $d = 0$ and $d = 1$. The case $d = 0$ corresponds to a type of natural resource: land; the case $d = 1$ corresponds to another type of natural resource: raw materials, for example, minerals.

The motion law of the natural resource is

$$S_{t+1} = G(S_t - dR_t); \quad \forall t \in \mathbb{N}.$$

Approach two. For any $t \in \mathbb{N}$, at the end of period $t - 1$, the natural resource, the stock of which is S_t, is owned by all the young men of period t, it grows to $G(S_t)$ at the beginning of period t, and the young men become old. Then, the old men sell it to the young men, the property rights of the natural resource is transfered from the old to the young. And then, the young men harvest a part of the natural resource, R_t, and sell it to the firms, the rest of stock, $G(S_t) - dR_t$, (where $d \in [0; 1]$ is the depreciation rate of the natural resource), is held by the young men at the end of period t. The motion law of the natural resource is

$$S_{t+1} = G(S_t) - dR_t; \quad \forall t \in \mathbb{N}.$$

Here, G, the regeneration function, is smooth, concave and nonnegative, defined on $[0; \infty)$, satisfying $G(0) = 0$, $G'(0) \in (0; \infty]$. An example is $G(x) = \;\; x^{\alpha} - \;\; x^{\beta}$, where $\; > 0; \;\; \geq 0; 0 < \;\; \leq 1 < \;\;$ are constants. In particular, $G(x) = x$.

The essential results under these two approaches are similar, in this paper, we take approach one[2].

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[1] Such a treatment is used in Tirole(1985), Rhee(1991), among others.

[2] Approach two is used in Farmer(2000), among others. Concerning the transaction of the natural resource, there are alternative approaches, for example, at each period, the old men harvest the resource and sell it to the firms, and sell the rest of the stock to the young men.

Decision making

All markets are completely competitive. We use the final good as numeraire, the price of which is normalized as 1. We denote the prices of the physical capital, labor, resource stock and resource harvest at time t as r_t, ω_t, p_t, q_t, respectively. And suppose that any young man has perfect foresight about all factors' prices in the next period.

At any time $t \in \mathbb{N}$, the decision problem for each young man is

$$\max_{(a,b,s,S,R)} U(a, b),$$

s.t.

$$\begin{aligned} a + s + p_t S/N_t &= \omega_t + (1 + q_t)R/N_t, \\ b &= (1 + r_{t+1})s + p_{t+1}G(S - dR)/N_t, \\ a \geq 0, \quad b &\geq 0, \quad s \geq 0, \quad 0 \leq R \leq S, \end{aligned}$$

where a, b are his consumption in this period and the next period, s is his savings for the investment in physical capital in the next period, S is the resource stock all the young men buy, R is the resource harvest sold to the firms.

At any time $t \in \mathbb{N}$, the decision problem for each firm is

$$\max_{(K,L,R)} \left\{ F^t(K, L, R) - (\delta_0(t) + r_t)K - \omega_t L - (d + q_t)R \right\},$$

where K, L, R are the physical capital, labor and resource harvest the firm buys, and the depreciation rate of capital is assumed to be $\delta_0(t) \in [0, 1]$.

Without any loss of generality, throughout this paper, we may assume that $\delta_0(t) = 1$ for any $t \in \mathbb{N}$, since otherwise, we replace $F^t(K, L, R)$ by $F^t(K, L, R) + (1 - \delta_0(t))$.

We say that this economy is in (dynamic) competitive equilibrium, if at any time, any firm has got its profit maximized, for the given prices of all related commodities, any individual has got his utility maximized, under his budget constraint, and all the markets are cleared.

More precisely, we define the dynamic equilibrium as follows.

Definition 1. A sequence of vectors $\{r_t, \omega_t, p_t, q_t, a_t, b_t, K_t, S_t, R_t\}_{t \in \mathbb{N}}$ is an equilibrium, if

(i) for any $t \in \mathbb{N}$,

$$(a_t, b_{t+1}, K_{t+1}/N_t, S_t, R_t) \in \arg\max_{(a,b,s,S,R)} U(a, b),$$

s.t.

$$\begin{aligned} a + s + p_t S/N_t &= \omega_t + (1 + q_t)R/N_t, \\ b &= (1 + r_{t+1})s + p_{t+1}G(S - dR)/N_t, \\ a \geq 0, \quad b &\geq 0, \quad s \geq 0, \quad 0 \leq R \leq S; \end{aligned}$$

and $N_{-1}b_0 = (1 + r_0)K_0 + p_0 S_0$;

(ii) for any $t \in \mathbb{N}$,

$$(K_t, N_t, R_t) \in \arg\max_{(K,L,R)} \left\{ F^t(K, L, R) - (1 + r_t)K - \omega_t L - (d + q_t)R \right\},$$

s.t.

$$K \geq 0; \quad L \geq 0; \quad R \geq 0;$$

(iii) for any $t \in \mathbb{N}$,

$$S_{t+1} = G(S_t - dR_t).$$

It's easy to verify that if $\{r_t, \omega_t, p_t, q_t, a_t, b_t, K_t, S_t, R_t\}_{t \in \mathbb{N}}$ is an equilibrium, then, for any $t \in \mathbb{N}$,

$$1 + r_t, \omega_t, p_t, d + q_t, a_t, b_t, K_t, S_t, R_t$$

are all positive.

In the rest of this paper, we mainly concern with the case $d = 1$. The case of land, corresponding to $d = 0, G(x) = x$, can be treated similarly.

Proposition 1. If $\{r_t, \omega_t, p_t, q_t, a_t, b_t, K_t, S_t, R_t\}_{t \in \mathbb{N}}$ is an equilibrium, then, for any $t \in \mathbb{N}$,

$$1 + r_t = F^t_K(K_t, N_t, R_t); \quad \omega_t = F^t_N(K_t, N_t, R_t); \quad p_t = F^t_R(K_t, N_t, R_t);$$

$$1 + q_t = p_t = \frac{p_{t+1} g'(S_t - R_t)}{1 + r_{t+1}}; \quad 1 + r_{t+1} = \frac{U_a(a_t, b_{t+1})}{U_b(a_t, b_{t+1})};$$

$$N_t a_t = Y_t - N_{t-1} b_t - K_{t+1}; \quad N_{t-1} b_t = (1 + r_t) K_t + p_t S_t;$$

where $Y_t = F^t(K_t, N_t, R_t)$.

The relationship

$$p_t = \frac{p_{t+1} G'(S_t - R_t)}{1 + r_{t+1}}, \quad \forall t \in \mathbb{N},$$

is the No-arbitrage condition, a generalized Hotelling rule, which is reduced to the classical Hotelling rule (HotellinG(1931)) when $G(x) = x$.

The relationship

$$F^{t+1}_K(K_{t+1}, N_{t+1}, R_{t+1}) = \frac{U_a(a_t, b_{t+1})}{U_b(a_t, b_{t+1})}$$

means that the MRS (of consumption today to that next day) is equal to the MRT (between savings today and the final good produced the next day).

From now on, we do not continue to use the notation q_t, and instead, we use $p_t - 1$ to replace it. In addition, we directly say that $\{r_t, \omega_t, p_t, a_t, b_t, K_t, S_t, R_t\}_{t \in \mathbb{N}}$ is an equilibrium, if $\{r_t, \omega_t, p_t, p_t - 1, a_t, b_t, K_t, S_t, R_t\}_{t \in \mathbb{N}}$ is an equilibrium in the above definition.

In this paper, we mainly concern the issue of Pareto efficiency of the equilibrium allocation. As to the existence of the equilibrium, this is another issue[1].

[1] For the existence of equilibrium in OLG model, see Balasko, Cass and shell (1980), or Geanakoplos(2008). Our main concern is the Pareto efficiency of the equilibrium allocation. In the case, where there are multiple equilibria, we take any one of them and investigate its efficient property.

condition for the solution. This approach will give us sufficiency conditions for the equilibrium allocation to be Pareto efficient. (See next section).

From Proposition 2, we can get a corollary, which is a necessary condition for the Pareto efficiency of any feasible allocation, which is, however, too weak. This is just the Euler equations in some sense, the basic idea of which can be stated as follows: the "optimal" path can not be improved locally in any segment of the path.

Corollary 1. If a feasible allocation $\{a_t, b_t, K_t, S_t, R_t\}_{t \in \mathbb{N}}$ is Pareto efficient, then, for any $t \in \mathbb{N}$,

$$\begin{aligned} & \frac{F_R^{t+1}(K_{t+1}, N_{t+1}, R_{t+1}) G'(S_t - R_t)}{F_R^t(K_t, N_t, R_t)} \\ = \quad & F_K^{t+1}(K_{t+1}, N_{t+1}, R_{t+1}) = \frac{U_a(a_t, b_{t+1})}{U_b(a_t, b_{t+1})}. \end{aligned}$$

By Proposition 1, we see that the equilibrium allocation satisfies this necessary condition, but, in general, it is not sufficient for the equilibrium allocation to be Pareto efficient, just as demonstrated in the classical Diamond two-period OLG model.

Corollary 1 can also be got as follows. It's just the first order conditions for the following problem: for any $t \in \mathbb{N}$, all other variables are given, find $(a_t, b_{t+1}, K_{t+1}, S_{t+1}, R_t, R_{t+1})$ to solve

$$\begin{aligned} \max \quad & U(a_t, b_{t+1}), \\ \text{s.t.} \quad & F^t(K_t, N_t, R_t) = N_t a_t + N_{t-1} b_t + K_{t+1}, \\ & F^{t+1}(K_{t+1}, N_{t+1}, R_{t+1}) = N_{t+1} a_{t+1} + N_t b_{t+1} + K_{t+2}, \\ & S_{t+1} = G(S_t - R_t), \\ & S_{t+2} = G(S_{t+1} - R_{t+1}). \end{aligned}$$

That is, fix all other variables (and hence the utilities of all other generations are fixed), and let $(a_t, b_{t+1}, K_{t+1}, S_{t+1}, R_t, R_{t+1})$ change, we can not increase the utility of the t- generation.

In the end of this section, we mention another relevant efficiency concept, for which we need to introduce Ramsey economy, which is related to but is in sharp contrast to the above OLG economy.

We "contract" the above OLG economy to the so-called Ramsey economy: there is one representative agent with infinite long life, and his endowment of labor (or human capital) is N_t at time t, and he owns the physical capital K_0 and natural resource S_0 in time 0, and the technologies are the same as above, and his consumption at time t is

$$C_t = N_t a_t + N_{t-1} b_t. \tag{1}$$

For this Ramsey economy, a sequence of nonnegative vectors $\{C_t, K_t, R_t, S_t\}_{t \in \mathbb{N}}$ is called feasible, if for any $t \in \mathbb{N}$,

$$K_{t+1} \le F^t(K_t, N_t, R_t) - C_t,$$

$$S_{t+1} = G(S_t - R_t).$$

A feasible allocation $\{C_t, K_t, S_t, R_t\}_{t\in\mathbb{N}}$ is called dynamically efficient, if there does not exist another feasible allocation $\{C'_t, K'_t, S'_t, R'_t\}_{t\in\mathbb{N}}$ ($K'_0 = K_0, S'_0 = S_0$) such that for any $t \in \mathbb{N}$,

$$C_t \le C'_t,$$

and at least one of the above inequalities holds strict inequality. A feasible allocation in this Ramsey economy is called dynamically inefficient, if it is not dynamically efficient.

For the Ramsey economy, the dynamic efficiency is studied in many works previously, such as Malinvaud sufficiency condition and Cass criterion, and their various extensions.

Now, return to the OLG economy. A feasible allocation $\{a_t, b_t, K_t, R_t, S_t\}_{t\in\mathbb{N}}$ in the above OLG economy is called aggregately efficient, if the corresponding feasible allocation $\{C_t, K_t, R_t, S_t\}_{t\in\mathbb{N}}$ in the above Ramsey economy is dynamically efficient, where C_t is defined by (1). It's aggregately inefficient, if it's not aggregately efficient.

Obviously, aggregate efficiency is weaker than Pareto efficiency. But, in general, the converse is not true[1]. An extreme counterexample: we change a Pareto efficient allocation to a new allocation by keeping the productions unchanged and giving all the consumptions of the old to the young at the same period, and suppose that the utility function for the t-generation is $U(a, b)$ satisfying $U(a, 0) = -\infty$ for any $a > 0$.

As to the utility functional of the representative agent in this Ramsey economy, in order to be consistent with the utility function in the above OLG economy, we consider a special case, where the utility function in the above OLG economy is

$$U(a, b) = u(a) + \beta u(b),$$

where $\beta \in (0, 1)$, and u is a smooth, strictly increasing and strictly concave function on $[0, \infty)$.

Take $\varepsilon \in (0, 1)$. Now, for any $t \in \mathbb{N}$, define a function on $[0, \infty)$:

$$\begin{aligned}\Phi_t(C) =: \ & \max \quad \varepsilon N_t u(a) + \beta N_{t-1} u(b), \\ & \text{s.t.} \quad N_t a + N_{t-1} b \le C.\end{aligned}$$

Then, we define the utility functional of the representative agent as

$$V_\varepsilon(\{C_t\}_{t\in\mathbb{N}}) = \sum_{t=0}^{\infty} \varepsilon^t \Phi_t(C_t),$$

where $\{C_t\}_{t\in\mathbb{N}}$ is the sequence of his consumptions at all times.

[1] But, if we only consider the equilibrium allocation, then, the problem whether these two concepts (Pareto efficiency and aggregate efficiency) are equivalent or not is still open.

For the above OLG economy, for any given $\varepsilon \in (0,1)$, we define a social welfare functional[1] on $\mathscr{A}$:

$$W_\varepsilon(\{a_t; b_t; K_t; S_t; R_t\}_{t\in\mathbb{N}}) = N_{1}u(b_0) + \sum_{t=0} \varepsilon^{t+1} N_t \left(u(a_t) + u(b_{t+1})\right).$$

Clearly, for any feasible allocation $\{a_t; b_t; K_t; R_t; S_t\}_{t\in\mathbb{N}}$ in $\mathscr{A}$, we have

$W_\varepsilon(\{a$

For any $t \geq 1$, denote the market discount factor from time t to time 0 as

$$D_t = \prod_{s=1}^{t} (1 + r_s)^{-1};$$

and $D_0 =: 1$. For any $t \in \mathbb{N}$, denote the sum of the individual investments of all the young men as

$$J_t = K_{t+1} + p_t(S_t - R_t);$$

and denote the total value of assets as

$$V_t = (1 + r_t)K_t + p_t S_t;$$

That is, at time t, the old men hold the assets $(K_t; S_t)$, through trade in the markets, they can get the total revenue V_t, and consume it. In fact, by Proposition 1, we have $V_t = N_{t-1}b_t$. And, at time t, the young men make a total investment J_t, including two parts: investment to physical capital K_{t+1} and investment to natural resource $p_t(S_t - R_t)$, and at time $t + 1$, they get old and get the total investment revenue V_{t+1}. Then, in this sense, we can say that such an investment earns a dividend

$$d_{t+1} = V_{t+1} - J_t = r_{t+1}K_{t+1} + p_{t+1}S_{t+1} - p_t(S_t - R_t):$$

For any $t \in \mathbb{N}$, denote the growth rate of investment to physical capital at time t and the growth rate of total investment at time t as

$$i_t = \frac{K_{t+1}}{K_t} - 1; \quad j_t = \frac{J_{t+1}}{J_t} - 1;$$

respectively.

It's worthwhile to point out that at any time $t \in \mathbb{N}$, the aggregate investment for the whole society (in our setting) is

$$K_{t+1} = Y_t - C_t;$$

which in fact is the total savings from human production, (rest of the total output, after subtracting the total consumption), saved for the usage in the production in the next period. But, we also notice that at the meantime, there are "savings" of another type: the savings from nature: $S_t - R_t$ at the end of t-period, which will grow to $S_{t+1} = G(S_t - R_t)$. In each period, the production depends, besides labor, not only on the savings from human production in the last period but also on the savings from nature in the last period.

This feature is not captured in AMSZ(1989), in which they assume that all the production factors come from the savings from human production in some previous periods, besides labor and some random perturbation. Therefore, AMSZ(1989) can not cover the case with natural resources.

To present sufficiency conditions and necessity conditions for the Pareto efficiency of the equilibrium allocation, AMSZ(1989) takes an approach in terms of d_t and J_t. And, just because of this, their criterion is called dividend criterion by some authors thereafter. In our model with natural resources, using d_t and J_t is not convenient. We will modify the approach in AMSZ(1989), and use K_t, V_t, instead.

In addition, the approach in Croix et al (2004) is also interesting, they study the efficiency issue by investigating the limit behavior of wages. We will modify their approach and use the total income in some sense.

In the sequel, we give two general sufficiency conditions and two general necessity conditions for the Pareto efficiency of the equilibrium allocation. The proofs can be found in Appendix.

3.1 Sufficiency condition for efficiency

Now, we state the general sufficiency conditions.

3.1.1 In terms of wage

Theorem 1. The equilibrium allocation is Pareto efficient, if

$$\liminf_{t\to\infty} D_t \omega_t N_t = 0. \tag{2}$$

The condition (2) is the same as in Lemma 2.1 in Croix et al (2004), in the standard Diamond OLG model, which is a modification of the condition in Theorem 1 in Homburg (1992). Our Theorem 1 says that the condition of Croix et al still works, even the economy is added natural resource. This condition means that there is nothing valuable left finally, in other words, roughly, all income from labor is used up finally.

Since along the equilibrium path, we have that for any $t \in \mathbb{N}$,

$$D_t p_t = D_{t+1} p_{t+1} G'(S_t - R_t);$$

then,

$$D_{t+1} p_{t+1} R_{t+1} = D_t p_t R_t \frac{R_{t+1}}{R_t G'(S_t - R_t)}.$$

Therefore, from Theorem 1, we get

Corollary 2. For the above equilibrium, if

$$\liminf_{t\to\infty} \frac{R_t G'(S_t - R_t)}{R_{t+1}} > 1; \tag{3}$$

$$\liminf_{t\to\infty} \frac{N_t F_N^t(K_t, N_t, R_t)}{R_t F_R^t(K_t, N_t, R_t)} < \infty; \tag{4}$$

then, the equilibrium allocation is Pareto efficient.

Here, (3) means roughly that the natural resource is not harvested too quickly; (4) means that in production, the resource share is not nil, comparing with the labor share, at least asymptotically along the equilibrium path, in other words, roughly, the natural resource is essential in production.

3.1.2 In terms of total value of assets

We have an alternative sufficiency condition.

Theorem 1'. The equilibrium allocation is Pareto efficient, if

$$\lim_{t \to \infty} D_t V_t = 0. \tag{5}$$

Condition (5) is a Malinvaud type sufficiency condition. In an one-sector Ramsey economy with exhaustible resource, Mitra(1978) proves that condition (5) is also necessary for a competitive program to be dynamically efficient, under the additional assumption that the resource share in production (the elasticity of output to resource) is bounded away from zero.

Roughly, condition (5) is stronger than condition (2), at least in the case, where for any $t \in \mathbb{N}$,

$$F^t(K; N; R) = A_t K^{\alpha} N^{\beta} R^{\gamma};$$

where $A_t > 0$, $\alpha; \beta; \gamma \in (0; 1)$, and $\alpha + \beta + \gamma = 1$. In addition, for application, condition (2) is more convenient than (5).

3.2 Sufficiency condition for inefficiency

As mentioned in section 2, the concept Pareto efficiency is stronger than the concept aggregate efficiency, then, all the relevant results about the dynamic inefficiency in Ramsey economy can be used here. More precisely, an allocation in the above OLG economy is Pareto inefficient, if the corresponding allocation in the "contracted" Ramsey economy is dynamic inefficient.

3.2.1 In terms of capital

Theorem 2. The equilibrium allocation is Pareto inefficient, if

$$\liminf_{t \to \infty} \frac{1 + i_t}{1 + r_t} > 1. \tag{6}$$

This is part of the popular view: if the growth rate of investment is higher than the rate of return of investment, then, the capital is over-accumulated, and this makes the economy inefficient.

The sufficient condition for Pareto inefficiency of the equilibrium allocation, given in AMSZ(1989) (without natural resources), is

$$\frac{1 + r_t}{1 + j_t} \quad \varepsilon; \quad \forall t \in \mathbb{N};$$

for some $\varepsilon \in (0; 1)$. But, in our setting with natural resources, such a condition can not guarantee the Pareto inefficiency, we modify it to (6), instead.

3.2.2 In terms of wage

We give an alternative sufficiency condition for the Pareto inefficiency of the equilibrium allocation in terms of wages for a special case, where the utility function is of CRRA type.

Theorem 2'. Suppose the utility function is

$$U(a,b) = u(a) + \beta u(b),$$

where $\beta \in (0,1)$, and

$$u(x) = \frac{x^{1-\sigma} - 1}{1-\sigma}, \quad x \geq 0,$$

where $\sigma \in (0,1]$. The equilibrium allocation is Pareto inefficient, if

$$\liminf_{t \to \infty} (1 + r_t)^{1-\sigma} > 0, \tag{7}$$

and

$$\liminf_{t \to \infty} \frac{\beta^{-1/\sigma+} + (1 + r_{t+1})^{(1-\sigma)/\sigma}}{\beta^{-1/\sigma+} + (1 + r_{t+2})^{(1-\sigma)/\sigma}} \frac{D_{t+1} I_{t+1} N_{t+1}}{D_t I_t N_t} > 1, \tag{8}$$

where for any $t \in \mathbb{N}$,

$$I_t = \omega_t + \frac{1}{N_t}\left(\frac{p_{t+1}}{1 + r_{t+1}} S_{t+1} - p_t(S_t - R_t)\right),$$

which indicates the total income of an individual of t-generation, composing of income from labor and income from investment to natural resource.

Condition (7) disappears, if $\sigma = 1$; it becomes

$$\liminf_{t \to \infty} r_t > -1,$$

if $\sigma \in (0,1)$.

Condition (8) means roughly that the present value of total income for t-generation is increasing finally. It's partially opposite to condition (2).

From Theorem 2', we can easily get a corollary.

Corollary 4. Under the assumptions of Theorem 2', if furthermore along the equilibrium path, the economy converges to a nontrivial steady state, in particular, as $t \to \infty$,

$$p_t \to p^* > 0, \quad \omega_t \to \omega^* > 0, \quad r_t \to r^* > -1,$$

$$S_t \to S^* > 0, \quad R_t \to R^* \in (0, S^*),$$

and

$$N_t = (1 + n)^t, \quad \forall t \in N,$$

where $n > -1$. Then, if $r^* > n$, then, the equilibrium allocation is Pareto efficient; if $r^* < n$, then, the equilibrium allocation is Pareto inefficient.

3.3 OLG with land

In the end of this section, we mention that a similar results holds for the case, where the natural resource is land, rather than the raw material considered above.

In fact, for the case of land, we suppose that there exists an equilibrium

$$\{r_t, \omega_t, p_t, q_t, a_t, b_t, K_t, S_t, R_t\}_{t \in \mathbb{N}},$$

where, in particular, p_t and q_t are the price of land and the rental of land at time t, respectively. All other variables have the meanings as above.

The transaction of land is as follows. For any $t \in \mathbb{N}$, at the beginning of period t, the old men sell the land to the young at price p_t, then, the young rent it to the firms at price q_t, and at the end of period t, the young hold the land to the next period[1].

By no-arbitrage condition, we have that for any $t \in \mathbb{N}$,

$$p_t - q_t = \frac{p_{t+1}}{1 + r_{t+1}},$$

then,

$$D_t p_t = D_{t+1} p_{t+1} + D_t q_t.$$

Therefore, $(D_t p_t)_{t \in \mathbb{N}}$ is decreasing, and hence, there is $\beta_0 \geq 0$ such that $D_t p_t \to \beta_0$ as $t \to \infty$, and for any $t \in \mathbb{N}$,

$$p_t = f_t + \beta_t,$$

where

$$f_t = \sum_{s=t}^{\infty} \frac{D_s}{D_t} q_s, \quad \beta_t = \beta_0 / D_t,$$

are the fundamental and the bubble of land at t, respectively. And,

$$p_0 = \beta_0 + \sum_{t=0}^{\infty} D_t q_t,$$

therefore,

$$\sum_{t=0}^{\infty} D_t q_t < \infty,$$

implying

$$\lim_{t \to \infty} D_t q_t = 0.$$

Now, we state the results concerning the efficiency (inefficiency) of the equilibrium allocation. The proofs are the same as in the case of raw material, hence, omitted.

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[1] It's a bit different from Rhee(1991) which allows the old to rent the land to the firms and then sell it to the young. And hence, here, p_t is the price of land before dividend, rather than the price of land after dividend in Rhee(1991).

In case of land, in principle, Theorems 1, 2, 1', 2' still hold, except that condition (5) in Theorem 1' can be weakened to

$$\lim_{t\to\infty} D_t K_{t+1} = 0;$$

and in Theorem 2', by the no-arbitrage condition, the income from the investment to land is zero, and hence, for any $t \in \mathbb{N}$, $I_t = \omega_t$.

Thus, if

$$\liminf_{t\to\infty} \frac{q_t}{\omega_t N_t} > 0; \tag{9}$$

then, condition (2) holds.

Here, condition (9) is a bit weaker than the sufficiency condition

$$\liminf_{t\to\infty} \frac{q_t}{Y_t} > 0;$$

(where Y_t is the total output of the consumption good at time t), which is given in Proposition 1 in Rhee (1991).

But, Rhee (1991) presents[1] an example that there is an equilibrium path (asymptotically bubbly), along which

$$\lim_{t\to\infty} D_t \omega_t N_t \in (0; \infty);$$

but the corresponding equilibrium allocation is Pareto efficient. It indicates that condition (2) is sufficient but not necessary for the equilibrium allocation to be Pareto efficient.

4 Linear regeneration function

In the basic setup in section 2, we make further assumptions as follows.

A1. For any $t \in \mathbb{N}$,

$$F^t(K; L; R) = A_t(\alpha K^\sigma + \beta L^\sigma + \gamma R^\sigma)^{1/\sigma};$$

where $\sigma < 1$, $0 < \alpha; \beta; \gamma < 1$ are given constants, satisfying $\alpha + \beta + \gamma = 1$; and for any $t \in \mathbb{N}$, $A_t > 0$ is a constant representing the TFP [2]. In particular, if $\sigma = 0$, then, F^t is reduced to Cobb-Douglas type: $F^t = A_t K_t^\alpha L_t^\beta R_t^\gamma$.

A2. For any $t \in \mathbb{N}$,

$$U(a_t; b_{t+1}) = \ln a_t + \delta \ln b_{t+1};$$

where $\delta \in (0; 1)$ is the discount factor. And anyone of the original generation has a utility function $\delta \ln b_0$, where b_0 is her consumption at $t = 0$. Or, put it another way, let's make a convention that $a_{-1} \equiv 1$.

[1] See Rhee (1991)(section III. A counterexample).

[2] It needs to be emphasized that in our setting, the condition $\gamma > 0$ is crucial, that is, the natural resource is essential in production. If $\gamma = 0$, then, the model will be reduced to the classical Diamond OLG model, in which the equilibrium allocation is not necessarily Pareto efficient.

A3. The motion law of the natural resource is

$$S_{t+1} = \quad (S_t \quad R_t);$$

where $\quad > 0$ is a constant, which can be called the strength of the regeneration of the resource. The exhaustible resource is our special case, where $\quad = 1$.

The concept of social optimality with respect to (wrt for short) some given social welfare functional is standard.

For any $\varepsilon \in (0,1)$, we construct a social welfare functional W_ε as follows: for any feasible allocation $\{a_t, b_t, K_t, S_t, R_t\}_{t \in \mathbb{N}}$, we define its social welfare value as the weighted sum of the utility values of all generations it induces:

$$\begin{aligned} & W_\varepsilon \quad \{a_t, b_t, K_t, S_t, R_t\}_{t \in \mathbb{N}} \\ = & \quad \ln b_0 + \varepsilon \sum_{t=0}^{\infty} (\varepsilon^t / N_t)(\ln a_t + \quad \ln b_{t+1}) N_t; \\ = & \sum_{t=0}^{\infty} \varepsilon^t (\varepsilon \ln a_t + \quad \ln b_t). \end{aligned}$$

Now, wa say a feasible allocation $\{a_t, b_t, K_t, S_t, R_t\}_{t \in \mathbb{N}}$ socially optimal wrt W_ε, if its social welfare value wrt W_ε is bigger than or equal to the social welfare value of any other feasible allocation wrt W_ε.

Obviously, if a feasible allocation is socially optimal wrt some social welfare functional W_ε with some $\varepsilon \in (0,1)$, then, it is Pareto efficient.

The special case, where $\quad = 1; \quad = 0$, is studied in Agnani et al. (2005).

4.1 Cobb-Douglas case

We first consider the case, where $\quad = 0$.

4.1.1 Equilibrium existence, uniqueness and efficiency

By Proposition 1 and Lemma 1 in Appendix, we obtain the following result.

Proposition 3. The equilibrium exists and is unique, which is determined uniquely by the conditions: for $\forall t \in \mathbb{N}$,

$$1 + r_t = \frac{\quad Y_t}{K_t}; \quad \omega_t = \frac{\quad Y_t}{N_t}; \quad p_t = \frac{\quad Y_t}{R_t};$$

$$N_t a_t = \frac{\quad}{1 + \quad} Y_t; \quad N_{t-1} b_t = \quad + \frac{\quad}{1 \quad} \quad Y_t;$$

$$K_{t+1} = \quad Y_t; \quad S_t = (\quad)^t S_0; \quad R_t = (1 \quad)(\quad)^t S_0;$$

where $Y_t = A_t K_t^\alpha N_t^\beta R_t^\gamma$, and

$$= \frac{(\quad + \quad + \quad) \quad \sqrt{(\quad + \quad + \quad)^2 \quad 4 \quad}}{2}; \tag{10}$$

where

$$= \frac{}{1 +}:$$

Clearly, in the case of = 0, by ignoring $p_t; S_t; R_t$, and replacing by = , this result is reduced to the equilibrium result in the classical Diamond model.

We see that in the equilibrium path, the natural resource planar dynamic di erence system $(S_t; R_t)_{t \in \mathbb{N}}$ does not concern with the technical progress, population growth and the capital at all.

As above, we denote the market discount factor from time t to time 0 by D_t, and market discount factor from time s t to time t by $D(t; s)$.

By Proposition 3, we see that for any t 1,

$$Y_t = \frac{1}{}(1 + r_t)K_t = (1 + r_t) \; Y_{t-1};$$

$$N_t \omega_t = \; Y_t; \quad p_t R_t = \; Y_t; \quad R_t = (1 \quad)S_t;$$

therefore,

$$D_t Y_t = {}^t Y_0; \quad \forall t \in \mathbb{N};$$

thus,

$$\sum_{t=0}^{\infty} D_t Y_t < \infty; \tag{11}$$

$$p_t S_t = \sum_{s=t}^{\infty} D(t; s) p_s R_s; \quad \forall t \in \mathbb{N}: \tag{12}$$

(11) means that the present value of the output ow is nite. (12) means that the natural resource has no bubble, that is, at any time, the value of the stock is equal to the value of the harvest ow from then on.

Clearly,

$$\lim_{t \to \infty} D_t \omega_t N_t = \lim_{t \to \infty} D_t Y_t = 0;$$

and hence, by Theorem 1, we get the following proposition.

Proposition 4. The equilibrium allocation is Pareto e cient.

That is, if only the regeneration function is linear, the production is of Cobb-Douglas, the utility function is additive log, and, most importantly, > 0, then, the equilibrium is always Pareto e cient. This is in sharp contrast to the classical Diamond OLG model without natural resource.

By the way, in this case, (3) and (4) are also satis ed.

4.1.2 Asymptotic property and social optimality

It's worthwhile to mention that so far, we do not make any assumption about the patterns of changes of technology A_t and population N_t. In this section, we make a further assumption.

A4. For any $t \in \mathbb{N}$,

$$A_t = (1 + \quad)^t; \quad N_t = (1 + n)^t;$$

where $\quad 0; n > \quad 1$ are constants.

For any $t \in \mathbb{N}$, let $U_t = \ln a_t + \quad \ln b_{t+1}$; which can be taken as the welfare measurement for the t-generation. (We make the convention $a_{\ 1} = 1$)

We now look at the asymptotic properties of $k_t; y_t; a_t; b_t; U_t; r_t; p_t; \omega_t$ as $t \to \infty$ along the equilibrium path, where $k_t = K_t/N_t; y_t = Y_t/N_t$.

From the Proposition 3, one can obtain the following corollary immediately.

Corollary 5. Along the equilibrium path, as $t \to \infty$,

$$K_t \to 1 + \quad ; \quad Y_t \to 1 + \quad ;$$

$$a_t \to (1 + \quad)/(1 + n); \quad b_t \to (1 + \quad)/(1 + n);$$

$$k_t \to (1 + \quad)/(1 + n); \quad y_t \to (1 + \quad)/(1 + n);$$

$$\omega_t \to (1 + \quad)/(1 + n); \quad p_t \to \frac{1 + \quad}{\quad};$$

$$r_t \to r =: \frac{1 + \quad}{\quad} \quad 1; \quad U_t \quad t(1 + \quad) \ln((1 + \quad)/(1 + n)) \to v;$$

where v is some constant, and $1 + \quad = \left((1 + \quad)(1 + n)^{\beta}(\quad)^{\gamma} \right)^{1/(\beta+\gamma)}$.

Remark 1 (BGP). When $k_t; y_t; a_t; b_t$ grow at the same growth rate, then, we say that the economy is on a balanced growth path, BGP, for short. When the growth rates of $k_t; y_t; a_t; b_t$ approach to the same rate asymptotically, then, we say that the economy is on a BGP asymptotically, or, on an asymptotic BGP. When this same rate is negative, then, we say that this economy per capita contracts asymptotically; when this same rate is zero, then, we say that this economy per capita is steady asymptotically; when this same rate is positive, then, we say that this economy per capita booms asymptotically.

Note that the limit growth rate of the investment (to physical capital) is $\quad$, and the limit interest rate (rate of return of investment to physical capital) is r, since $\quad \in (0; 1)$, then, it always holds that $r > \quad$. Therefore, it does not occur overaccumulation of capital.

Form Corollary 5, we have immediately the following results.

Proposition 5. From any initial state $(K_0; S_0)$, the economy goes on a BGP asymptotically. The economy is on an exact BGP, if and only if

$$K_0^{\beta+\gamma} = \quad ((1 \quad)S_0)^{\gamma}.$$

Agnani et al. (2005) assume not only the existence and uniqueness of the equilibrium but also that the equilibrium path is an exact BGP. Our Proposition 4 indicates that the exact BGP is really rare to happen.

Denote

$$\quad = \quad (1 + \quad)^{1/\gamma} \quad (1 + n).$$

For fixed $\quad \in (0; 1); \quad > 0; \quad 0; n > \quad 1, \quad$ is a function of $(\ ;\ ;\ )$. The simplex

$$A = \quad (\ ;\ ;\ ) \ 0 < \ ;\ ;\ < 1; \ + \ + \ = 1 \tag{13}$$

can be divided to three areas, according to $< 0; \quad = 0; \quad > 0$, respectively. On A, $= 0$ is a continuous curve, the shape of which is roughly delineated in Agnani et al. (2005).

Proposition 6. The economy per capita contracts asymptotically; is steady asymptotically; booms asymptotically, if $< 0; \quad = 0; \quad > 0$, respectively.

Because is not the initial parameter of the model, it's better to induce from Proposition 5 some other formula in terms of the initial parameters.

It's easy to see that

$$\frac{}{+ \quad +} < \quad < \frac{}{+} :$$

Therefore, we have

Corollary 6. The economy per capita contracts asymptotically, if

$$\frac{(1 + \quad)^{1/\gamma}}{1 + n} \quad 1 \quad < \frac{1 +}{} \quad ;$$

the economy per capita booms asymptotically, if

$$\frac{(1 + \quad)^{1/\gamma}}{1 + n} + 1 \quad > 1 + \quad :$$

That is, if the population grows too quickly (comparing with natural resource regeneration rate and the technical change rate), or the labor share is too small, or, the discount factor is too small, then, the economy per capita contracts, and moreover, the wage goes to zero. Corollary 6 covers Proposition 2 in Agnani et al (2005) for exhaustible resource.

On the contrary, if technological progress is su ciently quick, the natural resource regenerates su ciently quickly, and the labor share is not so small, then, the economy per capita booms asymptotically, the wages go to in nity.

Moreover, we can get the social optimality of the equilibrium allocation, which is stronger than the Pareto e ciency. Recall is de ned in (10).

Proposition 7. The equilibrium allocation is socially optimal with respect to W_δ.

The proof can be found in Appendix.

4.2 CES case

Now, we consider the case of $\neq 0$. We restrict our consideration to the special case without technical progress and with exponential population growth, that is, $A_t = 1$, $N_t = (1 + n)^t$ for any $t \in \mathbb{N}$, where $n \quad 0$.

Analogous to the above analysis, we can get the dynamical system of the equilibrium path:

$$K_{t+1} = (\quad K_t^\sigma + \quad N_t^\sigma + \quad R_t^\sigma)^{(1 \quad \sigma)/\sigma} \quad N_t^\sigma + \quad R_t^\sigma \quad 1 \quad \frac{S_t}{R_t} \quad ;$$

$$R_{t+1} = \quad - ^{1/(1 \quad \sigma)} \frac{R_t}{K_t^\sigma + \quad N_t^\sigma + \quad R_t^\sigma} \quad N_t^\sigma + \quad R_t^\sigma \quad 1 \quad \frac{S_t}{R_t} \quad ;$$

$$S_{t+1} = \quad (S_t \quad R_t).$$

Denote the capital, resource stock and resource harvest per capita respectively as

$$k_t = \frac{K_t}{N_t}; \quad s_t = \frac{S_t}{N_t}; \quad z_t = \frac{R_t}{N_t}.$$

Then, we get the dynamical system of $(k_t; s_t; z_t)_{t \in \mathbb{N}}$:

$$k_{t+1} = \frac{1}{1+n} (\quad k_t^\sigma + \quad + \quad z_t^\sigma)^{(1 \quad \sigma)/\sigma} \quad + \quad z_t^\sigma \quad 1 \quad \frac{s_t}{z_t} \quad ; \tag{14}$$

$$z_{t+1} = \frac{1}{1+n} \quad - ^{1/(1 \quad \sigma)} \frac{z_t}{k_t^\sigma + \quad + \quad z_t^\sigma} \quad + \quad z_t^\sigma \quad 1 \quad \frac{s_t}{z_t} \quad ; \tag{15}$$

$$s_{t+1} = \frac{}{1+n} (s_t \quad z_t). \tag{16}$$

Any path $(k_t; s_t; z_t)_{t \in \mathbb{N}}$ satisfying the feasibility condition that for any $t \in \mathbb{N}$, $s_t > 0$, $z_t > 0$, and

$$0 < \frac{s_t}{z_t^{1 \quad \sigma}} \quad z_t^\sigma < \quad = \quad ;$$

induces an equilibrium. The corresponding prices of capital, labor and resource are

$$r_t = \quad k_t^{\sigma \quad 1} (\quad k_t^\sigma + \quad + \quad z_t^\sigma)^{(1 \quad \sigma)/\sigma} \quad 1; \tag{17}$$

$$\omega_t = \quad (\quad k_t^\sigma + \quad + \quad z_t^\sigma)^{(1 \quad \sigma)/\sigma}; \tag{18}$$

$$p_t = \quad z_t^{\sigma \quad 1} (\quad k_t^\sigma + \quad + \quad z_t^\sigma)^{(1 \quad \sigma)/\sigma}; \tag{19}$$

respectively. And, by the Hotelling rule, for any $t \in \mathbb{N}$, the income from investment to the natural resource is zero, and hence, the total income in Theorem 2' is only the income from labor, that is, $I_t = \omega_t$.

A feasible path $(k_t; s_t; z_t)_{t \in \mathbb{N}}$, together with its corresponding $(r_t; \omega_t; p_t)_{t \in \mathbb{N}}$, satisfying (14)-(19), is called a steady state equilibrium, if there exist $\hat{k}; \hat{s}; \hat{z} \in [0; \infty)$ such that as $t \to \infty$, $k_t \to \hat{k}$, $s_t \to \hat{s}$, $z_t \to \hat{z}$. If $\hat{s} > 0$, $\hat{z} > 0$, then, we say it's a nontrivial steady state equilibrium. If $\hat{s} = 0$, $\hat{z} = 0$, then, we say it's a trivial steady state equilibrium.

First of all, we consider the nontrivial steady state equilibrium. If there is nontrivial steady state equilibrium, then, by dropping o all subscripts in (14), (15) and (16), we can get a unique (closed form) solution of this steady state, from which we can conclude that there exists (unique) nontrivial steady state equilibrium, if and only if

$$> (1+n) \_ \quad \left[((1+n)(1+ \quad) = \quad)^{1 \quad \sigma} \right]; \quad \quad ^\sigma > \quad .$$

And, correspondingly, the limit wage exists and is positive, and the limit interest rate is

$$\lim_{t \to \infty} r_t = \quad \quad 1.$$

And hence, by Theorem 1, this nontrivial equilibrium allocation is Pareto efficient.

Moreover, if > 0, then, there is a nontrivial steady state equilibrium, if and only if

$$> (1+n) \vee \left[((1+n)(1+\ )/\ )^{1-\sigma} \vee \ \right]^{1/\sigma}.$$

If < 0, then, there is a nontrivial steady state equilibrium, if and only if

$$^{1/\sigma} > \ > (1+n) \vee \left[((1+n)(1+\ )/\ )^{1-\sigma} \right],$$

which yields

$$< ((1+n)(1+\ )/\ )^{\sigma}.$$

That is, roughly, in the case, where the factors in production are substitutable, there exists a nontrivial steady state equilibrium, iff the regeneration rate of the natural resource is sufficiently large; while in the case, where the factors in production are complementary, there exists a nontrivial steady state equilibrium, iff the regeneration rate of the natural resource is located in some interval, in addition, is sufficiently small. (Here, be seen as a modified capital share in some sense.)

In the sequel, we consider the trivial steady state equilibria. We discuss it in two cases separately.

4.2.1 $\in (0, 1)$

It's easy to see that there are a continuum of trivial steady state equilibria: for any $\in [0, \ ]$, there is a steady state equilibrium such that as $t \to \infty$,

$$s_t \to 0, \quad z_t \to 0, \quad \frac{s_t}{z_t^{1-\sigma}} \to (\ \ )/\ , \quad k_t \to k,$$

where k can be determined uniquely by letting $t \to \infty$ in (14), that is,

$$k = \frac{}{1+n}(\ k^{\sigma} + \ )^{(1-\sigma)/\sigma}. \tag{20}$$

Correspondingly, as $t \to \infty$, $r_t \to r$, $\omega_t \to \omega$, where

$$1 + r = \ k^{\sigma-1}(\ k^{\sigma} + \ )^{(1-\sigma)/\sigma}, \tag{21}$$

$$\omega = \ (\ k^{\sigma} + \ )^{(1-\sigma)/\sigma} \in (0, \infty).$$

For $= 0$, correspondingly, $k = 0$, $r = \infty$, then, as $t \to \infty$,

$$D_t \omega_t N_t \to 0.$$

Therefore, by Theorem 1, the corresponding equilibrium allocation is Pareto efficient.

For any 2 (0;], by (20) and (21), the limit interest rate and the limit capital per capita of the corresponding trivial steady state equilibrium satisfy

$$\frac{1+r}{1+n} = \frac{k^{\sigma}}{\quad} =: b.$$

By Theorem 1 and Theorem 2 or Theorem 2', we have that if $b > 1$, then, the corresponding trivial steady state equilibrium allocation is Pareto e cient; if $b < 1$, then, it is Pareto ine cient[1].

Form (20), we get that b satis es

$$\frac{(1+n)^{\sigma}}{\quad} b = \quad b + \frac{}{\quad}^{1 \quad \sigma}.$$

And hence, $b > (=;<)1$, if and only if $< (=;>)^{-}$, where

$$^{-} =: \min \left(; \frac{}{\left[((1+n)^{\sigma} = \quad)^{1/(1 \quad \sigma)} \quad 1 \right]_{+}} \right);$$

where $[\,]_{+}$ indicates the positive part. Denote

$$\overline{n} = \left[(2 + 1 = \quad))^{1 \quad \sigma} \right]^{1/\sigma} \quad 1.$$

Clearly, $^{-} =$, if and only if $n \quad \overline{n}$.

Therefore, if $n < \overline{n}$, then, any trivial steady state equilibrium allocations are Pareto e cient. If $n \quad \overline{n}$, then, the allocations of any trivial steady state equilibria, corresponding to 2 $[0;^{-})$, are Pareto e cient; the allocations of any trivial steady state equilibria, corresponding to 2 $(^{-}; \,]$, are Pareto ine cient.

Noticing that

$$\lim_{t \,!\, 1} \frac{z_t}{s_t^{1/(1 \quad \sigma)}} = \quad \frac{}{\quad}^{1/(1 \quad \sigma)};$$

we can interpret as some indicator of speed of resource harvesting. Therefore, among the trivial steady state equilibria, concerning the resource harvesting, the slower, the better in the sense that it's more prone to be Pareto e cient.

4.2.2   < 0

It's easy to see that there are a continuum of trivial steady state equilibria: for any 2 [0;], there is a steady state equilibrium such that as $t \,!\, 1$,

$$s_t \,!\, 0; \quad z_t \,!\, 0; \quad \frac{s_t \quad z_t}{z_t^{1 \quad \sigma}} \,!\, (\qquad) = \,; \quad k_t \,!\, 0.$$

[1] The case $b = 1$ can not be treated directly by our Theorems. But we guess that it's Pareto e cient in that case. Then, that will be a counterexample for the necessity of condition (2).

Notice that as $t \to \infty$,

$$\frac{1 + r_t}{1 + i_t} = \frac{1 + r_t}{(1 + n)k_t}$$

As in the linear case, the planar di erence dynamical system of $(S_t; R_t)_{t2\mathbb{N}}$, described by (22) and (23), is determined only by the parameters $\;;\;;\;;\;$ and the regeneration function g, but does not concern with the capital and the parameters $\;; n$, that is, the capital, the technical progress and the population growth have no impact on the movement of the natural resource. And, R_0 is not determined for this moment.

By the generalized Hotelling rule, we know that along any equilibrium path, for any $t \; 2 \; \mathbb{N}$, we have

$$G^0(S_t \quad R_t) > 0;$$

then, for any $t \; 2 \; \mathbb{N}$,

$$S_t \quad R_t < a{=}b:$$

In addition, from $0 < R_{t+1} < S_{t+1}$, we get that for any $t \; 2 \; \mathbb{N}$,

$$< \quad [(\quad + \quad) R_t \qquad S_t] \frac{G^0(S_t \quad R_t)}{G(S_t \quad R_t)} < \quad + \quad (1 + \quad);$$

which, in particular, implies that for any $t \; 2 \; \mathbb{N}$,

$$R_t > \frac{}{\quad +} S_t:$$

Therefore, any path, going outside the area in the S-R plane:

$$(S; R) j (S \quad a{=}b)^+ \_ \; (\; {=}(\quad + \quad))S < R < S \quad ;$$

can not be an equilibrium path. And, any R_0, which induces a path, satisfying (22), (23) and (24), will give an equilibrium.

The planar dynamical system of $(S_t; R_t)_{t2\mathbb{N}}$ has possibly two steady states[1]: the rst one is the trivial $(0; 0)$[2]; the second is a non-trivial one $(S \;; R \;)$, where

$$R \quad = G(x \;) \quad x \;;$$
$$S \quad = G(x \;);$$

where $x \quad 2 \; (0; a{=}b)$ is determined by

$$+ \frac{}{1 +} \quad (2a \quad bx \;) \qquad = \frac{2}{1 +} (a \quad bx \;) \;\; 2a \quad 1 \quad - \quad bx \quad ;$$

which has a unique root in $(0; a{=}b)$.

About the more detailed location of x , we have the following Lemma, the proof is easy, hence, omitted.

Lemma 2. If $\quad < \quad$, then, $x \; < (a \quad 1{=}2){=}b$; if $\quad = \quad$, then, $x \; = (a \quad 1{=}2){=}b$; if $\quad > \quad$, then, $x \; > (a \quad 1{=}2){=}b$, where

$$= \frac{1 +}{} \quad + \quad \frac{1}{-} + \frac{1 +}{(a \quad 1{=}2)} \quad ;$$

[1] In the linear case, there is only one steady state, which is the trivial one $(0; 0)$. But here, if a is not su ciently large, the nontrivial steady state may not exist.

[2] In this state, $a_t; b_t; K_t; Y_t$ are all zero, and the economy collapses.

which is a function of $;$ for xed $;a$. can be seen as an aggregate index of physical capital share and natural capital share , for simplicity, it is called aggregate capital index.

Then, the simplex A, de ned in (13), can be divided to three areas, corresponding to $<$, $=$, $>$, respectively.

It's easy to verify that for this dynamical system, the Jacobian matrix at the steady state $(0;0)$ has two eigenvalues bigger than 1, and hence, the steady state $(0;0)$ is a source, to which no feasible path converges.

Now the second steady state $(S\ ;R\ )$. Denote the Jacobian matrix at the steady state $(S\ ;R\ )$ as $\mathbb{A}$, denote the trace of $\mathbb{A}$ as T, the determinant of $\mathbb{A}$ as D. Then, the characteristic equation for $\mathbb{A}$ is

$$x^2 \quad Tx + D = 0;$$

where x is the eigenvalue of $\mathbb{A}$. Noticing $a{=}b$ is su ciently large, one can verify directly that

$$T > 0; \quad D > 0; \quad T^2 > 4D; \quad 1 + D < T;$$

and hence, $\mathbb{A}$ has two positive eigenvalues, one is smaller than 1, the other is bigger than 1. Therefore, $(S\ ;R\ )$ is a saddle.

Consequently, there exists a unique R_0 which induces a unique saddle path converging to this saddle, and then, the unique equilibrium follows.

Now, we discuss the e ciency of the equilibrium allocation and the issue of sustainability of this economy. To this end, we rst give a corollary from the above analysis. The proof can be found in Appendix.

Denote

$$k_t = \frac{K_t}{N_t}; \quad y_t = \frac{Y_t}{N_t}; \quad {}_t = D_t I_t N_t:$$

and

$$g =: \ (1 + \ )(1 + n)^{\beta} \ ^{1/(\beta+\gamma)} \quad 1;$$

$$r =: G^0(x\ )(1 + g) \quad 1; \quad ' := \ \frac{1 + }{(1 + n)^{\gamma}} \ ^{1/(\beta+\gamma)} :$$

Corollary 7. Along the equilibrium path, as $t\ !\ 1$,

$$K_t \ ! \ 1 + g; \quad Y_t \ ! \ 1 + g;$$

$$k_t \ ! \ '; \quad y_t \ ! \ '; \quad a_t \ ! \ '; \quad b_t \ ! \ ';$$

$$r_t \ ! \ r; \quad !_t \ ! \ '; \quad p_t \ ! \ 1 + g; \quad {}_t \ ! \ \frac{1}{G^0(x\ )}:$$

We see that g is the asymptotic growth rate of K_t and Y_t, and r is the asymptotic interest rate. And, as $t\ !\ 1$, $a_t; b_t; k_t; y_t$ all grow at the same rate asymptotically. In other words, the economy goes on an asymptotic BGP.

Concerning the Pareto e ciency, we know that if $<$, then, by Lemma 2, $g^0(x\ ) > 1$, and hence, from Corollary 7, $r > g$. Therefore, (2) is satis ed. Thus, by Theorem 1, the equilibrium allocation is Pareto e cient.

If > , then, by Lemma 2, $g^0(x\ ) < 1$, and hence, from Corollary 7, $r < g$. Thus, by Theorem 2, the equilibrium allocation is Pareto ine cient.

To sum up, we get[1]

Proposition 8. If < , then, the equilibrium allocation is Pareto e cient; if > , then, the equilibrium allocation is Pareto ine cient.

We see that the labor share plays crucial role. If the labor share is smaller than the aggregate capital index, then, e cient; if the labor share is bigger than the aggregate capital index, then, ine cient. This is not like the linear case above, where the equilibrium allocation is always Pareto e cient, no matter how the factor shares are distributed, and how high or low the growth rate of the resource.

On the simplex A, the line segment = has two endpoints, the coordinates of them are $(\bar{\ };\bar{\ };0)$ and $(0;\underline{\ };\underline{\ })$, respectively, where

$$\bar{\ } = \frac{}{1+2};\quad \bar{\ } = \frac{1+}{1+2};$$

$$\underline{\ } = \frac{1+2a+2}{(1+\ )(1+2a)};\quad \underline{\ } = \frac{(2a\quad 1)}{(1+\ )(1+2a)}:$$

Recall a is su ciently large, then $\underline{\ } < \bar{\ }$.

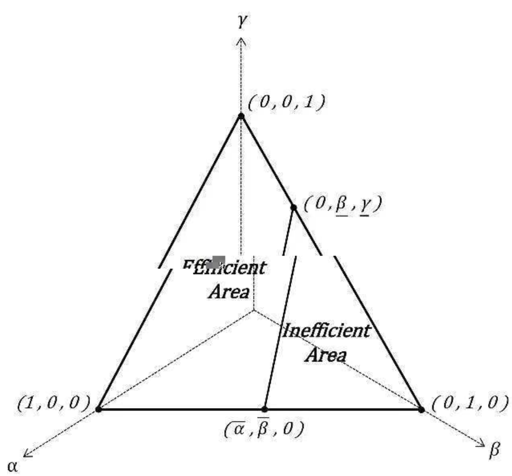

Figure 1: Division of Simplex A.

Clearly, if $> \bar{\ }$, or $< \underline{\ }$, then, < ; if $> \underline{\ }$, then, > . And, of course, if $> \underline{\ }$, then, $< \underline{\ }$. Then, from Proposition 8, we get

Corollary 8. If $> \bar{\ }$, or $< \underline{\ }$, then, the equilibrium allocation is Pareto e cient; if $> \bar{\ }$, then, the equilibrium allocation is Pareto ine cient.

[1] As to the cut-edge case = , the issue becomes quite tough, even in the classical Diamond OLG model. The rigorous mathematical treatment is di cult.

From this corollary, we see that if the labor share is small enough (including the case, where the resource share is large enough), or the capital share is large enough, then, Pareto e ciency holds; if the labor share is large enough, then, Pareto ine ciency holds, no matter how the capital share and resource share are distributed. If the labor share is in the medium, and the capital share is not so large, then, it may or may not be Pareto e cient, if the resource share is relatively small, (roughly, labor is not much important in production), then, e ciency holds; if it is relatively large, (roughly, labor is quite much important in production), then, ine ciency holds.

Because for each factor, the factor income share is the index of its intensity in the production, then, we can say roughly that if the technology is capital-intensive (either the physical capital or the natural capital), then the economy is e cient; on the contrary, if the technology is labor-intensive, then, the economy is ine cient.

A particular case, included in the above general setting, is that there is neither technical growth nor population growth. In this case, it's easy to see that K_t will converge to some positive level, denoted as K . When = , then, $K = K_{GR}$, where K_{GR} is the so-called Golden rule level of capital; when < , then, $K < K_{GR}$, and the economy is e cient; when > , then, $K > K_{GR}$, that is, it occurs the capital over-accumulation, the economy is ine cient.

Concerning the sustainability, from Corollary 6, we have

Proposition 9. If $1 + \; < (1+n)^{\gamma}$, then, the economy per capita contracts asymptotically; if $1 + \; = (1+n)^{\gamma}$, then, the economy is sustainable in the long run; if $1 + \; > (1+n)^{\gamma}$, then the economy per capita booms asymptotically.

That is, that the economy (per capita) contracts or not depends only on the technical progress rate, population growth rate and the resource share, it has nothing to do with the distribution of capital share and labor share. Of course, here, we have already made the basic assumption that the resource has a su ciently large intrinsic growth rate and carrying capacity of the environment for it.

6 Conclusion

We consider a two-period OLG model with three factors of production: physical capital, labor and natural resource. We discuss the issue of Pareto e ciency of the equilibrium allocation.

Our main contribution to the literature is that we present general su ciency conditions and general necessary conditions for the Pareto e ciency of the equilibrium allocation in the OLG economies with natural resources. Among the related previous works, the seminal paper AMSZ(1989) can not cover the case with natural resources, and Koskela, et al (2002), an important paper, does not consider physical capital. Then, our general results are applied to several concrete cases.

A minor contribution is that for the case, where the regeneration function of the natural resource is linear, the utility function is additive log function, the production function is of Cobb-Douglas, we give a rigorous proof for the existence and uniqueness of the equilibrium and the Pareto e ciency of the equilibrium allocation without any additional assumptions on technical progress and population change and the economic development path (e.g. the assumption of BGP).

Another contribution is that we nd when the regeneration function of the natural resource is linear, the utility function is additive log function, the production function is of general CES (rather than Cobb-Douglas), there exist a continuum of trivial steady state equilibria, besides a (unique) possible nontrivial steady state equilibrium, which is Pareto e cient. If the factors are complementary, then, all the trivial steady state equilibrium allocations are Pareto e cient. If the factors are substitutable, then, some are Pareto e cient, some are not, depending on the speed of harvesting, the slower, the more prone to be Pareto e cient.

Still another contribution is that we nd when the regeneration function is quadratic (with additive log utility function), the unique equilibrium allocation may or may not be Pareto e cient, according to the distribution of the income share of the three factors. Roughly, if the technology is capital-intensive (either the physical capital or the natural capital), then the economy is e cient; on the contrary, if the technology is labor-intensive, then, the economy is ine cient.

To sum up the applications in the two of the above examples: utility function is additive log function, production function is of Cobb-Douglas, but the natural resource regeneration function is either linear or quadratic, we nd that when the resource regeneration function is linear, then, the economy is always Pareto e cient, the ine ciency issue disappears; when the resource regeneration function is quadratic, then, the ine ciency issue emerges again, just returning to the classical Diamond OLG economy, and, roughly, the phenomenon is almost the same as in Diamond, that is, when the industry is capital-intensive, then, e cient; when the industry is labor-intensive, then, the economy is ine cient.

Finally, we pose some problems to study further. One is a conjecture that under some assumptions on the production function and the utility function, the equilibrium allocation is Pareto e cient, if and only if

$$\sum_{t=1}^{\infty} \frac{1}{D_t \omega_t N_t} = \infty.$$

Another problem is to extend our study to multi-sector OLG models with natural resources.

Still another problem is to study the optimal intervention when the economy is Pareto ine cient.

A further problem is to study the e ciency issue at the scenario that the technology is endogenous, e.g., it can be chosen from a set of available technologies.

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Appendix

The proof of proposition 3 needs the following Lemma 1. A proof of Lemma 1 can be found in Farmer et al (2010), which uses the eigenvalues method in the planar dynamical di erence system. Here, we present another proof, which has its own interest.

Lemma 1. The following two statements about $fS_t; R_t g_{t2\mathbb{N}}$ are equivalent: (I) for any $t\ 2\ \mathbb{N}$,

$$S_{t+1} = \quad (S_t \quad R_t) \quad 0;$$
$$R_{t+1} = - [(\quad + \quad) R_t \quad\quad S_t] \quad 0;$$

(II) for any $t\ 2\ \mathbb{N}$,

$$S_t = (\quad)^t S_0;$$
$$R_t = \quad (\quad)^t S_0:$$

Proof. One can easily verify that (II) implies (I). In the sequel, we prove that (I) implies (II). First of all, we show that for any $t\ 2\ \mathbb{N}$ and any $n\ 2\ \mathbb{N}$, it holds that

$$x_n S_t \quad R_t \quad y_n S_t; \tag{25}$$

where

$$x_{n+1} = \frac{\quad + \quad x_n}{\quad + \quad + \quad x_n}; \quad x_0 = 0;$$
$$y_{n+1} = \frac{\quad + \quad y_n}{\quad + \quad + \quad y_n}; \quad y_0 = 1:$$

We prove (25) by use of the method of mathematical induction wrt n. First, obviously, (25) holds for $n = 0$ and any $t\ 2\ \mathbb{N}$. Now, suppose that (25) holds for n and any $t\ 2\ \mathbb{N}$. Then, for any $t\ 2\ \mathbb{N}$, notice that (25) holds for n and $t + 1$, that is,

$$x_n S_{t+1} \quad R_{t+1} \quad y_n S_{t+1};$$

which is equivalent to

$$x_{n+1} S_t \quad R_t \quad y_{n+1} S_t;$$

and hence, (25) also holds for $n + 1$ and any $t\ 2\ \mathbb{N}$. It follows that (25) holds for any $t\ 2\ \mathbb{N}$ and any $n\ 2\ \mathbb{N}$.

Next, clearly, $fx_n g_{n2\mathbb{N}}$ is increasing and bounded above, and $fy_n g_{n2\mathbb{N}}$ is decreasing and bounded below, and hence, each of these two sequences has limit, and obviously, their limits are the same, denoted as z, which satis es that

$$z = \frac{\quad + \quad z}{\quad + \quad + \quad z};$$

noticing that $z\ 2\ (0; 1)$, and hence, we get that $z = \quad$, where is de ned above.

Consequently, for any $t\ 2\ \mathbb{N}$,

$$R_t = \quad S_t;$$

which yields (I) immediately. The proof is completed.

Proof of Theorem 1. Suppose the equilibrium allocation is not Pareto efficient. Then, there is a feasible allocation $(a'_t, b'_t, K'_t, S'_t, R'_t)_{t\in\mathbb{N}}$, which is a Pareto improvement of the equilibrium allocation, therefore,

$$N_{-1}b'_0 \geq (1+r_0)K_0 + p_0 S_0, \tag{26}$$

$$N_t a'_t + \frac{N_t b'_{t+1}}{1+r_{t+1}} \geq N_t \omega_t + \frac{p_{t+1}}{1+r_{t+1}} S'_{t+1} - p_t(S'_t - R'_t), \quad \forall t \in \mathbb{N}, \tag{27}$$

and at least one of these inequalities holds strict inequality. Clearly, (27) is equivalent to

$$\begin{aligned} & D_t N_t a'_t + D_{t+1} N_t b'_{t+1} \\ \geq\ & D_t \omega_t N_t + D_{t+1} p_{t+1} S'_{t+1} - D_t p_t S'_t + D_t p_t R'_t, \quad \forall t \in \mathbb{N}. \end{aligned} \tag{28}$$

Since for any $t \in \mathbb{N}$, the maximum profit for any firm is 0, and noticing the condition of feasibility, then, we have

$$\begin{aligned} & (1+r_t)K'_t + \omega_t N_t + p_t R'_t \\ \geq\ & F^t(K'_t, N_t, R'_t) \\ \geq\ & N_t a'_t + N_{t-1} b'_t + K'_{t+1}, \end{aligned}$$

therefore,

$$\begin{aligned} & (1+r_0)K_0 + \omega_0 N_0 + p_0 R'_0 \\ \geq\ & N_0 a'_0 + N_{-1} b'_0 + K'_1, \end{aligned} \tag{29}$$

and for any $t \in \mathbb{N}$,

$$\begin{aligned} & D_t K'_{t+1} + D_{t+1}\omega_{t+1} N_{t+1} + D_{t+1} p_{t+1} R'_{t+1} \\ \geq\ & D_{t+1} N_{t+1} a'_{t+1} + D_{t+1} N_t b'_{t+1} + D_{t+1} K'_{t+2}. \end{aligned} \tag{30}$$

Since at least one of the inequalities in (26) and (28) has the strict inequality, then, there is $\epsilon > 0$ such that for sufficiently large T, summing up (26), (28), (29) and (30) for $t = 0$ through $t = T - 1$, we get

$$D_T \omega_T N_T > \epsilon + D_T(N_T a'_T + K'_{T+1}) + D_T p_T (S'_T - R'_T) \geq \epsilon.$$

Consequently,

$$\liminf_{t\to\infty} D_t \omega_t N_t > 0.$$

A contradiction. And hence, the equilibrium allocation is efficient. The proof is completed.

Proof of Theorem 1'. It suffices to show that the equilibrium allocation is the solution of the optimization problem in Proposition 2. To this end, put the Lagrangian

$$L = u(b_0)$$

$$+ \sum_{t=0}^{\infty} {}_t(F^t(K$$

$$\lim_{t \to \infty} D_t V_t = 0.$$

By Proposition 1, we get our desired results. The proof is completed.

Proof of Theorem 2. The spirit of the proof is the same as in AMSZ(1989). In fact, if (6) holds, then, we can construct a Pareto improvement of the equilibrium allocation. To this end, notice that for any $t \in \mathbb{N}$,

$$C_t + K_{t+1} = F^t(K_t, N_t, R_t),$$

where C_t is the total consumption at time t, defined in (1).

Now, fix $(N_t, R_t)_{t \in \mathbb{N}}$. If we fix $(C_t)_{t \geq 1}$ and make K_1 decreasing a bit, then, accordingly, C_0 will increase strictly, and for any $t \geq 1$, K_{t+1}, as a function of K_1, is strictly increasing, and by the chain rule, we can get that the elasticity of K_{t+1} with respect to K_1

$$\frac{dK_{t+1}/dK_1}{K_{t+1}/K_1} = \prod_{s=1} \frac{dK_{s+1}/dK_s}{K_{s+1}/K_s} = \prod_{s=1} \frac{1 + r_s}{1 + i_s}$$

is bounded uniformly and decreasing to 0 exponentially as $t \to \infty$, because of (6). Thus, such a construction of Pareto improvement is feasible. The proof is completed.

Proof of Theorem 2'. For any $t \in \mathbb{N}$, by solving the individual problem for the t-generation, we get

$$a_t = \frac{1}{1 + (\ (1 + r_{t+1})^{1\ \sigma})^{1/\sigma}} I_t,$$

$$b_{t+1} = \frac{(\ (1 + r_{t+1}))^{1/\sigma}}{1 + (\ (1 + r_{t+1})^{1\ \sigma})^{1/\sigma}} I_t.$$

For any $t \in \mathbb{N}$, consider the function of $\ \in [0, 1]$:

$$f_t(\ ) = u(a_t(1\ \ )) + \ u\left(b_{t+1} + \frac{N_{t+1}}{N_t} a_{t+1}\ \right).$$

It's easy to see that $f_t(\ )$ is strictly increasing in $[0,\ _t]$, where

$$_t = \frac{1\ \ _t^{\ 1/\sigma}}{1 + \ ^{1/\sigma}(\ _t(1 + r_{t+1}))^{1\ 1/\sigma}},$$

where

$$_t = \frac{\ ^{1/\sigma +} + (1 + r_{t+1})^{(1\ \sigma)/\sigma}}{\ ^{1/\sigma +} + (1 + r_{t+2})^{(1\ \sigma)/\sigma}} \frac{D_{t+1} I_{t+1} N_{t+1}}{D_t I_t N_t}.$$

By (7) and (8), we know that there exist $\ \in (0, 1)$ and $T \in \mathbb{N}$ such that for any $t \geq T$,

$$_t > \ .$$

Then, for any $t \quad T$,

$$f_t(\quad) > f_t(0).$$

Therefore, the feasible allocation $(a'_t, b'_t, K_t, S_t, R_t)_{t \in \mathbb{N}}$ is a Pareto improvement of the equilibrium allocation $(a_t, b_t, K_t, S_t, R_t)_{t \in \mathbb{N}}$, where for any $t < T$,

$$a'_t = a_t; \quad b'_t = b_t;$$

and for any $t \quad T$,

$$a'_t = a_t(1 \quad); \quad b'_t = b_t + \frac{N_t}{N_{t \ 1}} a_t \quad .$$

The proof is completed.

Proof of Proposition 4. The social planner's problem ($\mathbb{P}$) is

$$\max \sum_{t=0}^{\infty} {}^t(\ \ln a_t + \ \ln b_t);$$
$$\text{s.t.} \quad K_{t+1} = A_t K_t^{\alpha} N_t^{\beta} R_t^{\gamma} \quad N_t a_t \quad N_{t \ 1} b_t; \quad \forall t \in \mathbb{N};$$
$$S_{t+1} = \ (S_t \quad R_t); \quad \forall t \in \mathbb{N};$$

and all variables are nonnegative, where K_0, S_0 are given. By transformation

$$X_t = \ {}^t K_t; \quad H_t = \ {}^{1/\gamma} R_t; \quad Z_t = \ {}^{1/\gamma} S_t;$$
$${}^{(t+1)} N_t a_t = \frac{}{+} c_t; \qquad {}^{(t+1)} N_{t \ 1} b_t = \frac{}{+} c_t;$$

where $\ = gn^{\beta} \ {}^{1/(1 \ \alpha)}$; ($\mathbb{P}$) can be reduced to ($\mathbb{P}'$):

$$\max \sum_{t=0}^{\infty} {}^t \ln c_t;$$
$$\text{s.t.} \quad X_{t+1} = X_t^{\alpha} H_t^{\gamma} \quad c_t; \quad \forall t \in \mathbb{N};$$
$$Z_{t+1} = \ (Z_t \quad H_t); \quad \forall t \in \mathbb{N};$$

and all variables are nonnegative, where X_0, Z_0 are given.

By the standard dynamic programming approach (see Stokey, Lucas, Prescott(1989)), we have that the unique optimal Markovian strategy for ($\mathbb{P}'$) is

$$c = (1 \quad) \ {}^{\gamma} X^{\alpha} Z^{\gamma}; \quad H = \ Z.$$

Then, the optimal Markovian strategy for ($\mathbb{P}$) is unique, and satisfies that for $\forall t \in \mathbb{N}$,

$$N_t a_t = \frac{}{1+} Y_t; \quad N_{t \ 1} b_t = \quad + \ - \ Y_t; \quad Y_t = A_t K_t^{\alpha} N_t^{\beta} R_t^{\gamma};$$

$$K_{t+1} = \quad Y_t; \quad S_t = (\quad)^t S_0; \quad R_t = \ (\quad)^t S_0;$$

which is just the equilibrium allocation. The theorem is proved.

Proof of Corollary 6. For any $t \in \mathbb{N}$